

FEFFERMAN–GRAHAM AMBIENT METRICS OF PATTERSON–WALKER METRICS

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ABSTRACT. Given an n -dimensional manifold N with an affine connection D , we show that the associated Patterson–Walker metric g on T^*N admits a global and explicit Fefferman–Graham ambient metric. This provides a new and large class of conformal structures which are generically not conformally Einstein but for which the ambient metric exists to all orders and can be realized in a natural and explicit way. In particular, it follows that Patterson–Walker metrics have vanishing Fefferman–Graham obstruction tensors. As an application of the concrete ambient metric realization we show in addition that Patterson–Walker metrics have vanishing Q-curvature.

1. INTRODUCTION AND MAIN RESULT

Given a signature (p, q) conformal structure $[g]$ on an $m = p + q$ dimensional manifold M , it was shown in seminal work by Charles Fefferman and C. Robin Graham (see [FG85, FG12]) that under specific conditions the conformal structure can be encoded equivalently as a signature $(p + 1, q + 1)$ pseudo-Riemannian metric (\mathbf{M}, \mathbf{g}) with vanishing Ricci curvature. This description has been fundamental in constructing and classifying conformal invariants (see e.g. [FG85, BEG94]) and for constructing and studying conformally invariant differential operators (see [Gra92, GJMS92]).

To build the Fefferman–Graham ambient metric for given local coordinates x on M , one first considers the ray bundle of metrics in the conformal class $[g]$, written as $\mathbb{R}_+ \times \mathbb{R}^m$ with coordinates (t, x) . The ambient space \mathbf{M} is obtained by adding a new transversal coordinate $\rho \in \mathbb{R}$, and then an *ansatz* for the Fefferman–Graham ambient metric \mathbf{g} is

$$\mathbf{g} = t^2 g_{ij}(x, \rho) dx^i \odot dx^j + 2\rho dt \odot dt + 2t dt \odot d\rho, \quad (1)$$

where $g = g_{ij}(x, 0) dx^i dx^j$ is a representative metric in the conformal class. It is directly visible from the formula that \mathbf{g} is homogeneous of degree 2 with respect to the *Euler field* $t\partial_t$ on \mathbf{M} .

To show existence of a Fefferman–Graham ambient metric \mathbf{g} for given g , the *ansatz* (1) determines an iterative procedure to determine $g_{ij}(x, \rho)$ as a Taylor series in ρ satisfying $\text{Ric}(\mathbf{g}) = 0$ to infinite order at $\rho = 0$. For m odd existence (and a natural version of uniqueness) of \mathbf{g} as an infinity-order series expansion in ρ is guaranteed for general $g_{ij}(x)$. For $m = 2n$

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even, the existence of an infinity order jet for $g_{ij}(x, \rho)$ with $\text{Ric}(\mathbf{g}) = 0$ asymptotically at $\rho = 0$ is obstructed at order n . Existence of $g_{ij}(x, \rho)$ as an infinity order series expansion in ρ with $\text{Ric}(\mathbf{g}) = 0$ asymptotically at $\rho = 0$ is then equivalent to vanishing of the Fefferman–Graham obstruction tensor \mathcal{O} , which is a conformal invariant. Existence of \mathbf{g} for $m = 2n$ even does not in general guarantee uniqueness.

Results which provide global Fefferman–Graham ambient metrics, where \mathbf{g} can then be constructed in a natural way from g and satisfies $\text{Ric}(\mathbf{g})$ globally and not just asymptotically at $\rho = 0$ are rare, both in the odd- and even-dimensional situation. A special instance where global ambient metrics can at least be shown to exist occurs for g real-analytic, and m either being odd or m even and with obstruction tensor \mathcal{O} of g vanishing. The simplest case of geometric origin for which one has global ambient metrics consists of locally conformally flat structures $(M, [g])$, where (\mathbf{M}, \mathbf{g}) exists and is unique up to diffeomorphisms (see [FG12], ch. 7). Another well known geometric case are conformal structures $(M, [g])$ which contain an Einstein metric g : If $\text{Ric}(g) = 2\lambda(m-1)g$, then \mathbf{g} on $\mathbb{R}_+ \times M \times \mathbb{R}$ can be written directly in terms of g as

$$\mathbf{g} = t^2(1 + \lambda\rho)^2 g + 2\rho dt \odot dt + 2tdt \odot d\rho. \quad (2)$$

In work by Thomas Leistner and Paweł Nurowski it was shown that the so called *pp-waves* admit global ambient metrics in the odd-dimensional case and under specific assumptions in the even-dimensional case, see [LN10]. Concrete and explicit ambient metrics for specific examples of families of conformal structures induced by generic 2-distributions on 5-manifolds and generic 3-distributions on 6 manifolds have been constructed in [Nur08, LN12, Wil14, ALN15].

The present article expands the class of metrics for which canonical ambient metrics exist globally to *Patterson–Walker metrics*: Given an affine connection D on an n -manifold N with $n \geq 2$, which is supposed to be torsion-free and to preserve a volume form, the *Patterson–Walker metric* g is a natural split-signature (n, n) metric on T^*N , see recent work [HSSTZ16] for historical background on Patterson–Walker metrics, references and a modern treatment. Our main result is:

Theorem 1. *Let D be a torsion-free affine connection on N which preserves a volume form. Denote local coordinates on N by x^A and the induced canonical fibre coordinates on T^*N by p_A . Let $\Gamma_A^C{}_B$ denote the Christoffel symbols of D . Let*

$$g = 2 \, dx^A \odot dp_A - 2 \, \Gamma_A^C{}_B p_C \, dx^A \odot dx^B \quad (3)$$

*be the Patterson–Walker metric induced on T^*N by D . Then, with Ric_{AB} the Ricci curvature of D ,*

$$\begin{aligned} \mathbf{g} = & 2\rho dt \odot dt + 2tdt \odot d\rho \\ & + t^2(2dx^A \odot dp_A - 2p_C \Gamma_A^C{}_B dx^A \odot dx^B + \frac{2\rho}{n-1} \text{Ric}_{AB} dx^A \odot dx^B) \end{aligned} \quad (4)$$

is a globally Ricci-flat Fefferman–Graham ambient metric for the conformal class $[g]$.

We note that as an immediate consequence of the existence of the ambient metric for $(M, [g])$, the conformally invariant Fefferman–Graham obstruction tensor \mathcal{O} associated to $[g]$ vanishes.

It is not difficult to check Ricci-flatness of (4) directly: Specifically, one employs formula (3.17) of [FG85], which is applicable to any ambient metric in normal form (1). The computation is then based on the following key facts: The Ricci curvature of the Patterson–Walker metric g is up to a constant multiple just the pullback of the Ricci curvature of D and this tensor and its covariant derivative are totally isotropic, see [HSSTZ16]. We note that formula (4) for \mathbf{g} says that the ambient metric is in fact linear in ρ and the iterative procedure determining the ambient metric stops after the first step.

A geometric proof of vanishing Ricci curvature is presented in the next section. This is based on a combination of the well known Patterson–Walker and Thomas cone constructions, both of which we recall.

2. GEOMETRIC CONSTRUCTION OF THE AMBIENT METRIC

The association $D \rightsquigarrow g$ generalizes to a natural association from projective to conformal structures. Recall that two affine connections D, D' on N are called projectively related or equivalent if they have the same geodesics as unparameterized curves, which is the case if and only if there exists a 1-form $\Upsilon \in \Omega^1(N)$ with

$$D'_X Y = D_X Y + \Upsilon(X)Y + \Upsilon(Y)X \quad (5)$$

for all $X, Y \in \mathfrak{X}(N)$. It is sufficient to restrict ourselves to *special* connections in a projective class, i.e., to those that preserve some volume form. For projective structures it is useful to employ a suitably scaled *projective density bundle* $\mathcal{E}_+(1)$, defined as the special case of *weight* $w = 1$ of $\mathcal{E}(w) := (\wedge^n TN)^{-\frac{w}{n+1}}$. Then a section $s : N \rightarrow \mathcal{E}_+(1)$ corresponds to a choice of a special affine connection D in the projective equivalence class $[D]$, and any $s' = e^f s$ corresponds to D' projectively related to D via (5) with $\Upsilon = df$. We define $M = T^*N(2)$ the (projectively) weighted co-tangent bundle of N . Then, as was shown in [HSSTZ16], two projectively related affine connections D, D' on N induce two conformally related metrics g, g' on M , and we therefore have a natural association $(N, [D]) \rightsquigarrow (M, [g])$.

The cone $\mathcal{C} := \mathcal{E}_+(1)$ carries the canonical and well known Ricci-flat *Thomas cone connection* ∇ , see [Tho34] or [ČS09], where the specific weight 1 (different from [ČS09]) is most convenient for our computations. We will need a local formula for ∇ : Let $s : N \rightarrow \mathcal{E}_+(1)$ be the scale corresponding to an affine connection $D \in [D]$, providing a trivialization $\mathcal{E}_+(1) \cong \mathbb{R}_+ \times N$ via $(x^0, x) \mapsto s(x)x^0$. In this trivialization the Thomas cone connection is given by

$$\nabla_X Y = D_X Y - \frac{1}{n-1} \text{Ric}(X, Y)Z, \quad \nabla Z = \text{id}_{T\mathcal{C}} \quad (6)$$

where $X, Y \in \mathfrak{X}(N)$ and $Z = x^0 \partial_{x^0}$ is the Euler field on \mathcal{C} . It is in fact easy to see directly from formula (6) that the thus defined affine connection ∇ on the Thomas cone \mathcal{C} is independent of the choice of scale and Ricci-flat.

Employing a local coordinate patch on N which induces coordinates x^A , y_A on the co-tangent bundle T^*N and coordinates x^0, x^A, y_A, y_0 on $T^*\mathcal{C} \cong \mathbb{R}_+ \times T^*N \times \mathbb{R}$, the Patterson–Walker metric \mathbf{g} associated to ∇ is

$$\begin{aligned} \mathbf{g} = & 2dx^A \odot dy_A + 2dx^0 \odot dy_0 - \frac{4}{x^0} y_B dx^0 \odot dx^B \\ & - 2y_C \Gamma_A^C{}^B dx^A \odot dx^B + 2 \frac{x^0 y_0}{n-1} \text{Ric}_{AB} dx^A \odot dx^B. \end{aligned} \quad (7)$$

Ricci-flatness of \mathbf{g} follows directly from Ricci-flatness of ∇ , see Theorem 2 of [HSSTZ16]. Via the change of coordinates $t = x^0$, $\rho = \frac{y_0}{x^0}$, $p_A = \frac{y_A}{(x^0)^2}$ the metric \mathbf{g} transforms to (4), which is the form of a Fefferman–Graham ambient metric (1). In particular this shows Theorem 1.

Remark 2.1. As a Patterson–Walker metric (\mathbf{M}, \mathbf{g}) carries a naturally induced homothety \mathbf{k} of degree 2, which takes the form $2p_A \partial_{p_A} + 2\rho \partial_\rho$. According to Lemma 5.1 of [HSSTZ16] the infinitesimal affine symmetry Z of ∇ lifts to a Killing field, which one computes as $t\partial_t - 2p_A \partial_{p_A} - 2\rho \partial_\rho$. In particular it follows that the Euler field $t\partial_t$ of the Fefferman–Graham ambient metric \mathbf{g} can be written as the sum of this Killing field and the homothety \mathbf{k} . $T\mathbf{M}$ carries the maximally isotropic $(n+1)$ -dimensional subspace spanned by $\{\partial_{p_A}, \partial_\rho\}$ which is preserved by ∇ . This subspace can be equivalently described by a ∇ -parallel pure spinor \mathbf{s} on \mathbf{M} . The ambient Killing field \mathbf{k} and the ambient parallel pure spinor \mathbf{s} correspond to a homothety k of g and a parallel pure spinor χ on M that belong to the characterizing objects of the Patterson–Walker metric g , see Theorem 1 of [HSSTZ16].

We conclude this section by summarizing the construction:

Theorem 2. *Given a projective structure $(N, [D])$ on an n -dimensional manifold N , the geometric constructions indicated in the following diagram commute:*

$$\begin{array}{ccc} (\mathcal{C}, \nabla) & \rightsquigarrow & (\mathbf{M}, \mathbf{g}) \\ \uparrow & & \uparrow \\ (N, [D]) & \rightsquigarrow & (M, [g]) \end{array}$$

In particular, the induced conformal structure $[g]$ admits a globally Ricci-flat Fefferman–Graham ambient metric \mathbf{g} which is itself a Patterson–Walker metric.

We remark here that, for generic $[D]$, the resulting conformal class $[g]$ does not contain an Einstein metric, see Theorem 2 of [HSSTZ16]. In particular, one obtains a large class of conformal structures which are not conformally Einstein but which admit globally Ricci-flat and explicit ambient metrics.

3. VANISHING Q -CURVATURE

The Q -curvature Q_g of a given metric g is a Riemannian scalar invariant with a particularly simple transformation law with respect to conformal change of metric. It has been introduced by T. Branson in [Bra93] and has been the subject of intense research in recent years, see e.g. [CEOY08] for an overview. Computation of Q -curvature is notoriously difficult, see e.g.

[GP03]. An explicit form of a Fefferman–Graham ambient metric \mathbf{g} for a given metric g allows a computation of Q_g . Using the fact that \mathbf{g} is actually a Patterson–Walker metric, this computation is particularly simple.

Theorem 3. *The Patterson–Walker metric g associated to a volume-preserving, torsion-free affine connection D has vanishing Q -curvature Q_g .*

Proof. We follow the computation method for Q_g from [FH03]: For this it is necessary to compute $-\Delta^n \log(t)$, where Δ is the ambient Laplacian on $\mathbf{M} = \mathbb{R}_+ \times T^*N \times \mathbb{R}$ associated to \mathbf{g} and $t : \mathbf{M} \rightarrow \mathbb{R}_+$ is the first coordinate. Restricting $-\Delta^n \log(t)$ to the cone $\mathbb{R}_+ \times T^*N \times \{0\}$ and evaluating at $t = 1$ yields Q_g (see [FH03]). To show that Q -curvature vanishes for g , it is in particular sufficient to show that $\Delta \log(t) = 0$. However, the function $t : \mathbf{M} \rightarrow \mathbb{R}_+$ is horizontal since it is just the pullback of the coordinate function $x^0 : \mathcal{E}_+(1) \rightarrow \mathbb{R}_+$ on the Thomas cone $\mathcal{C} \cong \mathbb{R}_+ \times N$ via the canonical projection $T^*\mathcal{C} \rightarrow \mathcal{C}$. It follows from the explicit formula for the Christoffel symbols of a Patterson–Walker metric that Δ vanishes on any horizontal function, see [HSSTZ16], section 2.1. Thus in particular $\Delta \log(t) = 0$, and then also $Q_g = 0$. \square

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